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一类三维慢变振荡器周期解的高阶摄动解

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(基础课部)

摘要

本文推广了[1]的结果，得到了一类三维慢变振荡器周期解高阶摄动解。

主题词：浑沌；周期解；奇摄动解

引　言

最近浑沌运动的研究引起广泛的重视，已有了大量数值与实验结果，解析方法也在发展。Holmes在[3]中将Melnikov方法应用于二维Hamilton周期扰动系统的浑沌研究中，得出一些结果与实验及计算结果对应很好。最近Holmes又将该方法推广到高维情况，在[2]中研究了一类三维慢变振荡器系统

$$\begin{cases} \dot{X}=f_1(x,\bar{y},z)+\varepsilon g_1(x,y,z) \\ \dot{Y}=f_2(x,\bar{y},z)+\varepsilon g(x,y,z) \\ \dot{Z}=\varepsilon f_3(x,\bar{y},z) \end{cases}$$

利用推广的Melnikov方法得出周期解存在的定理，但是解的具体形式是不能给出的。我们在[1]中已给出了一类三维慢变振荡器系统的周期解的零阶摄动解，本文给出了高阶摄动解。满足我们条件的这类系统相当广泛，其中包括著名的Lorenz方程与Holmes在[2]中所有例子，因此我们给出的高阶摄动解是有一定的实际意义的。

1 周期解的高阶奇摄动解

我们研究三维慢变振荡器

$$\begin{cases} \dot{X}=f_1(x,\bar{y},z)+\varepsilon g_1(x,\bar{y},z) \\ \dot{Y}=f_2(x,\bar{y},z)+\varepsilon g_2(x,\bar{y},z) \\ \dot{Z}=\varepsilon f_3(x,\bar{y},z) \end{cases} \tag{1}$$

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这里$0<\varepsilon\ll 1$是小参数。$\varepsilon=0$时(1)成为

$$\begin{cases}\dot{X}=f_1(x,y,z)\\ \dot{Y}=f_2(x,y,z)\\ \dot{Z}=0\end{cases}\tag{2}$$

我们假设以下条件成立：

f_1、f_2、g_1、g_2是充分光滑的函数，在有界集上有界；

$\begin{pmatrix}f_1\\ f_2\end{pmatrix}$是Hamiloni向量场，即存在函数H使得$f_1=\partial H/\partial y$，$f_2=-\partial H/\partial x$。

今设(1)存在周期T的周期轨道，且设

$$\begin{cases}X=F+X\\ Y=G+Y\\ Z=H+Z\end{cases}\tag{3}$$

其中F、G、H是x,y,z的常数部分，X、Y、Z是振荡部分，即

$$F=\frac{1}{T}\int_0^T x\mathrm{d}t \qquad G=\frac{1}{T}\int_0^T y\mathrm{d}t \qquad H=\frac{1}{T}\int_0^T z\mathrm{d}t$$

将(3)代入(1)得

$$\begin{cases}\dot{X}=f_1(F+X,G+Y,H+Z)+\varepsilon g_1(F+X,G+Y,H+Z)\\ \dot{Y}=f_2(F+X,G+Y,H+Z)+\varepsilon g_2(F+X,G+Y,H+Z)\\ \dot{Z}=\varepsilon f_3(F+X,G+Y,H+Z)\end{cases}\tag{4}$$

由消去长期项的要求，(4)右边对t在一个周期中的平均必等于零，即

$$\begin{cases}\overline{f_1(F+X,G+Y,H+Z)+g_1\varepsilon_1(F+X,G+Y,H+Z)}=0\\ \overline{f_2(F+X,G+Y,H+Z)+g_2\varepsilon_2(F+X,G+Y,H+Z)}=0\\ \overline{f_3(F+X,G+Y,H+Z)}=0\end{cases}\tag{5}$$

我们用"—"表示平均，用"～"表示振荡部分。于是

$$\begin{cases}\dot{X}=\tilde{f_1}(F+X,G+Y,H+Z)+\varepsilon\tilde{g_1}(F+X,\ G+Y,H+Z)\\ \dot{Y}=f_2(F+X,G+Y,H+Z)+\varepsilon\tilde{g_2}(F+X,\ G+Y,H+Z)\\ \dot{Z}=\varepsilon f_3(F+X,G+Y,H+Z)\end{cases}\tag{6}$$

令

$$\begin{cases} X=\sum_{i=0}^{N} X_i\varepsilon^i \\ Y=\sum_{i=0}^{N} Y_i\varepsilon^i \\ Z=\sum_{i=0}^{N} Z_i\varepsilon^i \end{cases} \tag{7}$$

$$t=s\ (1+\sum_{i=0}^{N} W_i\varepsilon^i)$$

将(7)代入(6)，比较ε同次幂的系数得到

$$\begin{cases} dx_0/ds=\tilde{f}_1(F+X_0,G+Y_0,H+Z_0) \\ dy_0/ds=\tilde{f}_2(F+X_0,G+Y_0,H+Z_0) \\ dz_0/ds=0 \end{cases} \tag{8}$$

$$\begin{cases} dx_1/ds=(\partial\tilde{f}_1/\partial x)x_1+(\partial\tilde{f}_1/\partial y)y_1+(\partial\tilde{f}_1/\partial z)z_1+\tilde{g}_1+W_1\tilde{f}_1 \\ dy_1/ds=(\partial\tilde{f}_2/\partial x)x_1+(\partial\tilde{f}_2/\partial y)y_1+(\partial\tilde{f}_2/\partial z)z_1+\tilde{g}_2+W_1\tilde{f}_2 \end{cases} \tag{9}$$

$$dz_1/ds=\tilde{f}_3$$

………………

$$\begin{cases} dx_i/ds=(\partial\tilde{f}_1/\partial x)x_i+(\partial\tilde{f}_1/\partial y)y_i+(\partial\tilde{f}_1/\partial z)z_i+F_i(x_{i-1},y_{i-1},z_{i-1},W_0\cdots W_{i-1}) \\ dy_i/ds=(\partial\tilde{f}_2/\partial x)x_i+(\partial\tilde{f}_2/\partial y)y_i+(\partial\tilde{f}_2/\partial z)z_i+G_i(x_{i-1},y_{i-1},z_{i-1},W_0\cdots W_{i-1}) \\ dz_i/ds=(\partial\tilde{f}_3/\partial x)x_{i-1}+(\partial\tilde{f}_3/\partial y)y_{i-1}+(\partial\tilde{f}_3/\partial z)z_{i-1}+H(x_{i-2},\cdots z_{i-2},W_0\cdots W_{i-1}) \end{cases} \tag{10}$$

注意，我们在以上写法中省略了 $\tilde{f}_1$，$\tilde{f}_2$，$\tilde{f}_3$ $\partial\tilde{f}_1/\partial x$……在 $F+X_0,G+Y_0$，$H+Z_0$处取值的表示，其中 F_i、G_i、H_i 前面 i—1 个关系式递推地决定的项。利用[1]的方法可以求得 (1) 的零阶解表达式，不过此时表达式中存在待定系数，即

$$\begin{cases} X_0=\tilde{x}_0(S,a,b,F,G,H) \\ Y_0=\tilde{x}_0(S,a,b,F,G,H) \\ Z_0=0 \end{cases} \tag{11}$$

$Z_0=0$由(5)而知。

再将(11)代入(9)得

$$\begin{cases} dx_1/ds=(\partial\tilde{f}_1/\partial x)x_1+(\partial\tilde{f}_1/\partial y)y_1+(\partial\tilde{f}_1/\partial z)z_1+\tilde{g}_1+W_1\tilde{f}_1 \\ dy_1/ds=(\partial\tilde{f}_2/\partial x)x_1+(\partial\tilde{f}_2/\partial y)y_1+(\partial\tilde{f}_2/\partial z)z_1+\tilde{g}_2+W_1\tilde{f}_2 \\ dz_1/ds=f_3 \end{cases} \tag{12}$$

从(12)第3式可求得

$$Z_1=\tilde{z}_1(s)=\int\tilde{f}_3\ (F+\tilde{x}_0\ ,G+\tilde{y}_0,H+\tilde{z}_0)ds \tag{13}$$

再将(13)代入(12)前两式，如果

$$\begin{vmatrix} \partial\tilde{f}_1/\partial x, & \partial\tilde{f}_1/\partial y \\ \partial\tilde{f}_2/\partial x, & \partial\tilde{f}_2/\partial y \end{vmatrix} \neq 0$$

则由消去长期项的要求可求得a,b,w_1,且能解出

$$X_1=\tilde{x}_1(S,F,G,H)$$

$$Y_1=\tilde{y}_1(S,F,G,H)$$

一般地，我们已求出$w_1\cdots w_{i-2},x_0\cdots x_{i-1},y_0\cdots y_{i-1},z_0,\cdots z_{i-1}$，将它们代入(10)，可先求出

$$Z_i=z_i(s)=\int\left[(\partial\tilde{f}_3/\partial x)x_{i-1}+(\partial\tilde{f}_3/\partial y)y_{i-1}+(\partial\tilde{f}_3/\partial z)\ z_{i-1}++H_i\right]ds \tag{14}$$

再将(14)代入(10)前两式，由消去长期项要求可得W_{i-1},然后求得x_i,y_i，最后递推求得(7)。

2.1 例子

例1 修正的Van der pol方程([2])

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x+z+\varepsilon(x_2y-\delta y) \\ \dot{z}=\varepsilon(r-z+y^2) \end{cases}$$

令 $x=F+X$, $y=G+Y$, $z=H+Z$代入(15)得

$$\begin{cases} \dot{X}=Y+G \\ \dot{Y}=-F-X+H+Z+\varepsilon[(F+X)^2(G+Y)-\delta(G+Y)] \\ \dot{Z}=\varepsilon[r-H-Z+(G+Y)^2] \end{cases} \tag{16}$$

常数部分应满足

$$\begin{cases} G = 0 \\ -F + H + \varepsilon FG + 2\varepsilon F\overline{X\,Y} + \varepsilon\overline{X^2Y} = 0 \\ r - H + G^2 + \overline{Y}^2 = 0 \end{cases} \tag{17}$$

振荡部分应满足

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = -X + Z + \varepsilon(F^2Y + 2F\widetilde{XY} + \widetilde{X^2Y} - \delta Y) \\ \dot{Z} = \varepsilon\ (-Z + \widetilde{Y}^2) \end{cases} \tag{18}$$

我们求以下形式的解

$$\begin{cases} X = \sum_{i=0}^{N} \varepsilon^i X_i \\ Y = \sum_{i=0}^{N} \varepsilon^i Y_i \\ Z = \sum_{i=0}^{N} \varepsilon^i Z_i \\ t = S(1 + \sum_{i=1}^{N-1} w_i \varepsilon^i) \end{cases} \tag{19}$$

代入(18)展开比较 ε 的系数得

$$\begin{cases} dX_0/ds = Y_0 \\ dY_0/ds = -x_0 + z_0 \\ dZ_0/ds = 0 \end{cases} \tag{20}$$

$$\begin{cases} dX_1/ds = y_1 + y_0 w_1 \\ dY_1/ds = -x_1 + z + (F^2 - \delta)y_0 + 2F\widetilde{x_0 y_0} + \widetilde{x_0{}^2 y_0} - w_1 x_0 + w_1 z_0 \\ dZ_1/ds = -z_0 + \widetilde{y_0{}^2} \end{cases} \tag{21}$$

$$\begin{cases} dX_2/ds = Y_2 + w_2 Y_0 + w_1 Y_1 \\ dY_2/ds = -X_2 + Z_2 - w_2 X_0 + w_2 Z_0 - w_1 X_1 + w_1 Z_1 + F^2 X + w_1 F^2 X_0 \\ \qquad + 2F(\widetilde{X_0 Y_1} + \widetilde{X_1 Y_0} + w_1 \widetilde{X_0 Y_0}) + \widetilde{X_0{}^2 Y_1} + X_0{}^2 Y_1 w_1 - \delta Y_1 - w_1 Y_0 \\ dZ_2/ds = (-Z_1 + 2Y_0 Y_1 - w_1 Z_0 + w_1 Y_0{}^2) \end{cases} \tag{22}$$

从(20)求得

$$\begin{cases} X_0 = a\sin s \\ Y_0 = a\cos s \\ Z_0 = 0 \end{cases} \tag{23}$$

从(21)求得

$$Z_1 = a^2/4\sin 2s \qquad w_1 = 0 \qquad a^2 = 4(\delta - F^2)$$

$$X_1 = A_1\sin 2s + A_2\cos 3s$$

$$Y_1 = 2A_1\cos 2s - 3A_2\sin 3s$$

其中

$$A_1 = -\tfrac{1}{3}(a^2/4 + Fa^2)$$

$$A_2 = a^3/32$$

从(22)求得

$$Z_2 = B_1\cos 2s + B_2\cos 4s + B_3\sin 3s + B_4\sin s$$

其中

$$B_1 = a^2/8 - 3a/2 \qquad B_2 = 3aA_2/4 \qquad B_3 = 2A_1a/3 \qquad B_4 = -2aA_1$$

$$W_2 = 1/2a(B_4 - FaA_1 + 3A_2a^2/4)/2a$$

$$X_2 = A_3\cos 2s + A_4\cos 4s + A_5\sin 3s + A_5\sin 2s + A_7\sin 5s$$

$$Y_2 = -2A_3\sin 2s - 4A_4\sin 4s + 3A_5\cos 3s + 2A_6\cos 2s + 5A_7\cos 5s - w_2a\cos 3s$$

其中

$$A_3 = -\tfrac{1}{3}(B_1 - 2FaA_2 + a^2A_1 - 2\delta A_1)$$

$$A_4 = -1/15(B_2 + 4FaA_2 + a^2F_1/2)$$

$$A_5 = -\tfrac{1}{8}(B_3 + F^2A_2 + 3FaA_1 - 3a^2A_2/2 + 3\delta A_2)$$

$$A_6 = -\tfrac{1}{4}F^2A_1$$

$$A_7 = -3A_2a^2/96$$

最后我们得到

$$\begin{cases} x = F + X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + 0(\varepsilon^3) \\ y = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + 0(\varepsilon^3) \\ z = H + \varepsilon Z_1 + \varepsilon^2 Z_2 + 0(\varepsilon^3) \\ w = 1 + \varepsilon^2 w_2 \qquad + 0(\varepsilon^3) \\ a = \sqrt{(\delta - F^2)} \\ r - H + \tfrac{1}{2}a_2 + \varepsilon^2(2A_1^2 + 9A_2^2/2 + 2A_3^2 + 8A_4^2 + 9A_5^2/2 + 2A_6^2 + 25A_7^2/2) + \\ \varepsilon^2(w_2a/2)^2 = 0 \\ F = H + \varepsilon^2(a^2/2) + 0(\varepsilon^3) \end{cases} \tag{24}$$

例 2 核自旋发生器模型的周期轨道(见[4])

$$\begin{cases} \dot{x} = -\beta x + y \\ \dot{y} = -x - \beta y(1 - kz) \qquad 0 < \beta \ll 1 \\ \dot{z} = \beta[\alpha(1 - z) - ky^2] \end{cases} \tag{25}$$

以下取β＝δ，令

$$\begin{cases} x = F + X \\ y = G + Y \\ z = H + Z \end{cases} \tag{26}$$

代入(25)得

$$G = 0 \qquad F = 0 \qquad -2H + \alpha - K\overline{Y}^2 = 0 \tag{27}$$

和

$$\begin{cases} \dot{X} = -\beta(F+X) + G + Y \\ \dot{Y} = -(F+X) - \beta(G+Y)(1-KH-KZ) \\ \dot{Z} = \beta[\alpha(1-H-Z) - K(G+Y)^2] \end{cases} \tag{28}$$

再令

$$\begin{cases} X = \sum_{i=0}^{N} \varepsilon^i \\ Y = \sum_{i=0}^{N} \varepsilon^i Y_i \\ Z = \sum_{i=0}^{N} \varepsilon^i Z_i \\ t = \delta(1 + \sum_{i=1}^{N-1} w_i \varepsilon^i) \end{cases} \tag{29}$$

将(29)代入(28)，比较ε的系数得

$$\begin{cases} dX_0/ds = Y_0 \\ dY_0/ds = -X_0 \\ dZ_0/ds = 0 \end{cases} \tag{30}$$

$$\begin{cases} dX_1/ds = Y_1 - X_0 + w_1 Y_0 \\ dY_1/ds = -X_1 - Y_0(1-KH) - w_1 X_0 \\ dZ_1/ds = -\alpha Z_0 - \widetilde{K} Y_0^2 \end{cases} \tag{31}$$

$$\begin{cases} dX_2/ds = Y_2 + w_2 X_0 \\ dY_2/ds = -X_2 - w_2 X_0 - Y_0(-KZ_1) \\ dZ_2/ds = -\alpha Y_1 \end{cases} \tag{32}$$

从(30)可得

$$\begin{cases} X_0 = a\cos \\ Y_0 = -a\sin s \\ Z_0 = 0 \end{cases}$$

从(31)可得

$$\begin{cases} Z_1 = Ka^2\sin 2s/B \\ w_1 = 0 \\ a(1-KH)+a=0 \\ X_1 = Y_1 = 0 \end{cases}$$

从(32)可得

$$Z_2 = 2Ka^2\cos 2s/16$$

$$w_2 = K^2a^2/16$$

$$X_2 = A_1\cos 3s$$

$$Y_2 = -3A_1\sin 3s$$

其中

$$A_1 = -K^2a^3/72$$

最后我们得

$$\begin{cases} x = a\cos s + \varepsilon^2 A_1\cos 35 + 0(\varepsilon^3) \\ y = -a\sin s - \varepsilon^2 3A_1\cos 3s + 0(\varepsilon^3) \\ z = 2/K + \varepsilon(Ka^2\sin 2s/8) + \varepsilon^2(\alpha ka^2\cos 2s/16) + 0(\varepsilon^5) \\ a^2 = 2(\alpha - 2\alpha/K)/K \\ w = 1 + \varepsilon^2(-K^2a^2/16) \end{cases} \tag{33}$$

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Higher Order Perturbation Solutions of Periodic Orbits for a Class of 3 Dimersional Slowly Varying Oscillators

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Abstract

In this paper a method is given for the study of periodic orbits for a class of 3-dimensional slowly varying oscillators.

Subjectwords: Chaos; Periodic solution; Singular perturbation solution